

INTERACTIONS BETWEEN INTERLEAVING HOLES IN A SEA OF UNIT RHOMBI

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ABSTRACT. Consider a family of collinear, equilateral triangular holes of any even side length lying within a sea of unit rhombi. The results presented below show that as the distance between the holes grows large, the interaction between them may be approximated, up to a multiplicative constant, by taking the exponential of the negative of the electrostatic energy of the system obtained by viewing the holes as a set of point charges, each with a signed magnitude given by a certain statistic. Furthermore it is shown that the interaction between a family of left pointing collinear triangular holes and a free boundary may be approximated (again up to some multiplicative constant) by taking the exponential of the negative of the electrostatic energy of the system obtained by considering the holes as a set of point charges and the boundary a straight equipotential conductor. These two differing systems of point charges can be related via the method of image charges, a well-known physical law that also surfaces in the following mathematical analysis of enumeration formulas that count tilings of certain regions of the plane by unit rhombi.

1. INTRODUCTION

Interactions between holes (or gaps) in two dimensional dimer systems were first considered by Fisher and Stephenson, whose seminal paper [11] examined three types of interaction: the interaction between two dimers; the interaction between two monomers; and the interaction between a dimer and a fixed boundary (that is, an edge or a corner). While this work focused exclusively on interactions between holes in the square lattice, Kenyon [15] later generalised the first of these interaction types to an arbitrary number of dimer gaps on both the square and hexagonal lattices. Kenyon, Okounkov, and Sheffield [16] then extended these results even further to include general bipartite planar lattices.

Interactions between non-dimer gaps on the hexagonal lattice (in particular gaps consisting of a pair of monomers) have been studied extensively by Ciucu [3][4][5][6], establishing therewith close (conjectural) analogies between such interactions and two dimensional electrostatic phenomena. More specifically, Ciucu conjectures that the asymptotic interaction between non-dimer holes in a two dimensional dimer system on the hexagonal lattice is, up to a multiplicative constant, inversely proportional to the product of the pairwise distances between the holes raised to some exponential power that is determined by each pair of holes. Such an interaction is said to be governed by Coulomb's law for electrostatics since it may be obtained by taking the exponential of the negative of the electrostatic energy of the two-dimensional system of physical charges obtained by considering the holes as point charges of a certain magnitude and sign. A more formal statement of this conjecture together with a discussion of the evidence in support of it may be found in Ciucu's excellent survey paper [7]. Although it has been shown to hold for a very general class of holes in dimer systems that are

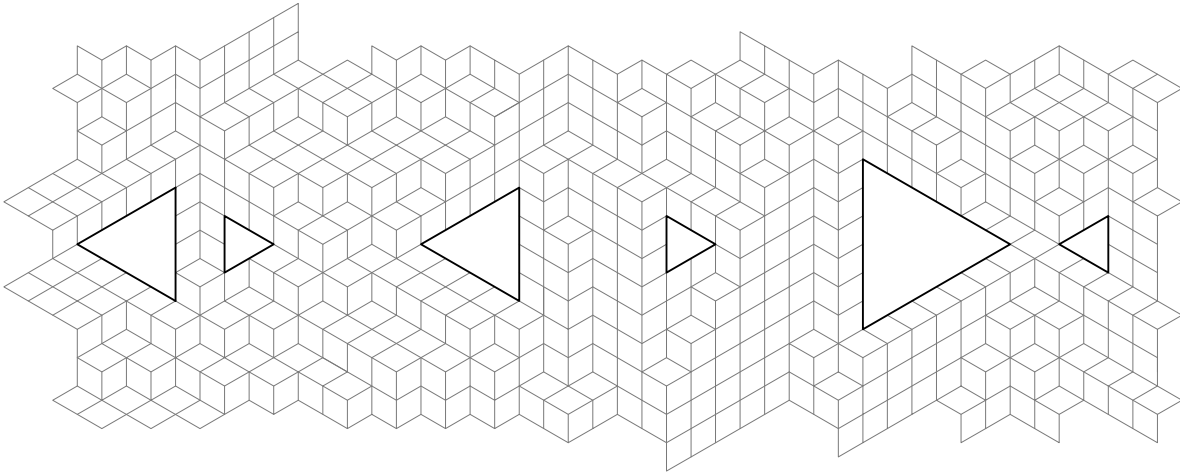


FIGURE 1. Part of a rhombus tiling of the plane, containing a set of horizontally collinear triangular holes of even side lengths where the sum of the charges of each hole is zero.

embedded on tori [5] and also for a certain class of holes in planar dimer systems [4], a complete proof of the conjecture remains elusive.

Further to the above mentioned interactions, a wholly (or perhaps, *hole-ly*) new type of non-dimer interaction was presented more recently in [9]: that of the interaction between a triangular hole and a so-called “free” boundary. Such an interaction also parallels certain physical phenomena, namely it appears to behave in analogy to the attraction of an electric charge to a straight line conductor. The author of the present work showed in [13] that similar behaviour may also be observed for a triangular hole that has been rotated 180° and thus points toward the boundary. Somewhat mysteriously, it seems the orientation of the hole has a direct effect on the interaction between the hole and the free boundary (a physical interpretation of this discrepancy has yet to be realised).

The contribution of the current article is twofold. Firstly, the interaction of an entirely new class of triangular holes is established, namely the interaction between *interleaving* holes of any even side length that lie along a horizontal line within a sea of unit rhombi (here interleaving means that all holes that point in one direction do not necessarily all lie to one side of all holes that point in the opposite direction, see Figure 1). It would appear that aside from the aforementioned results obtained by embedding tilings on tori [5], such interactions for the planar case have yet to be treated in the literature. Theorem 1 below shows that under certain conditions the planar case does indeed agree with Ciucu’s tori result, however the results stated in Section 4 show that this type of interaction depends somewhat delicately on the rate at which the boundaries of the plane approach infinity.

Secondly, the interaction between a set of left pointing triangular holes and a free boundary is established, thus generalising the earlier work of Ciucu and Krattenthaler [9] to include any (finite) number of left pointing triangular holes of any even side length. Once again it is shown that such interactions are in some sense governed by Coulomb’s law, since they may be approximated by taking the exponential of the negative of the electrostatic energy of the system obtained by viewing the holes as a set

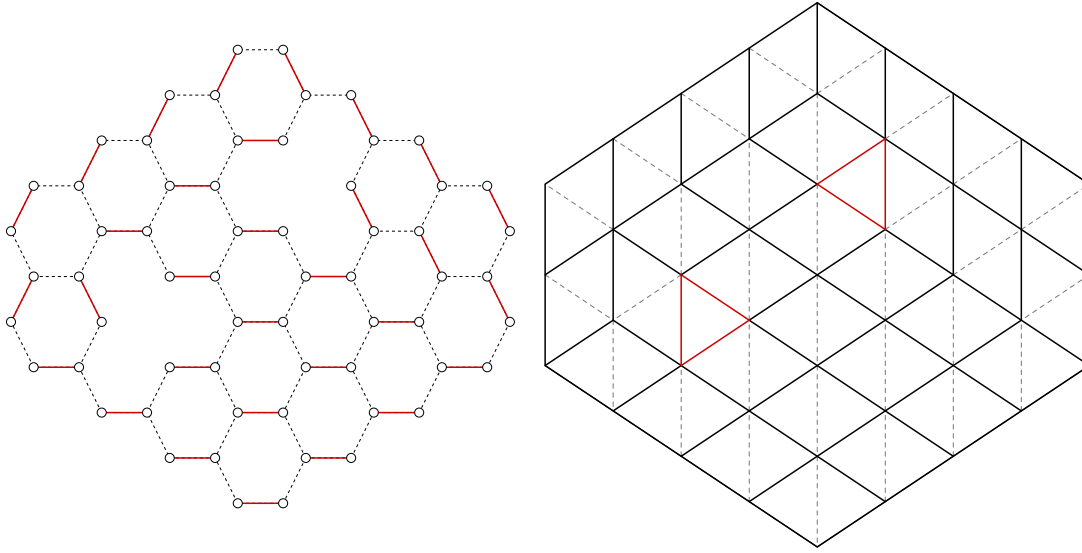


FIGURE 2. A dimer covering of a holey subregion of the hexagonal lattice, left, and the corresponding rhombus tiling on the triangular lattice, right. The unit triangles in red on the right correspond to the vertices that have been removed from the interior of the subgraph on the left.

of point charges and the free boundary a straight equipotential conductor. According to the method of images [10, Chapter 6] the electrostatic energy of such a system is half the electrostatic energy of the system obtained by replacing the conductor with imaginary charges (these charges of opposite signed magnitude are obtained by reflecting the original charges through the conductor). One sees this well-known physical law surfacing through the purely mathematical analysis of certain enumeration formulas that count tilings of certain regions of the plane by unit rhombi, thus adding further support to the on-going electrostatic program of Ciucu.

2. SET-UP AND RESULTS

The main results presented in this article concern *interactions* between holes in two dimensional *dimer systems*. In the spirit of Fisher and Stephenson [11], suppose \mathcal{R}_n is a subgraph (with size parametrised by n) of some two dimensional lattice with a fixed set of vertices (indexed by the set \mathcal{H}) removed from its interior. Denote this region containing a set of holes $\mathcal{R}_n \setminus \mathcal{H}$. A perfect matching between vertices in $\mathcal{R}_n \setminus \mathcal{H}$ is also known as a *dimer covering* of $\mathcal{R}_n \setminus \mathcal{H}$, and the set of all coverings of this region is known as a *dimer system*. As n tends to infinity, dimer coverings of $\mathcal{R}_n \setminus \mathcal{H}$ become dimer coverings of the entire plane (in other words, sending n to infinity yields a set of holes that lie within a sea of dimers, see Figure 1), and the *interaction* between the fixed holes (otherwise known as the *correlation function* of the holes) in the dimer system is defined to be

$$\omega_{\mathcal{R}}(\mathcal{H}) = \lim_{n \rightarrow \infty} \frac{M(\mathcal{R}_n \setminus \mathcal{H})}{M(\mathcal{R}_n)}, \quad (2.1)$$

where $M(\mathcal{R})$ denotes the number of dimer coverings (equivalently, perfect matchings) of the region \mathcal{R} . This paper focuses on dimer systems on the planar hexagonal lattice,

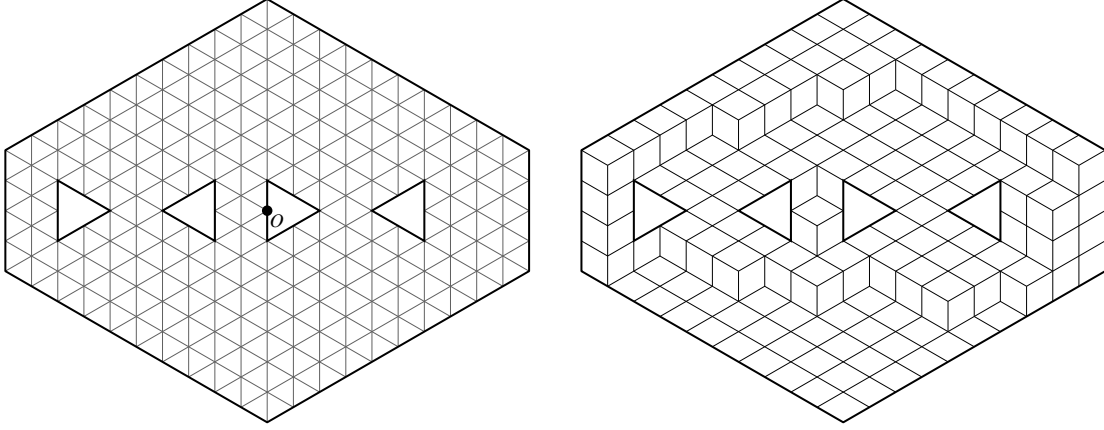


FIGURE 3. The holey hexagon $H_{10,4}^{\{-2,6\},\{0,-8\}}$, left, and a rhombus tiling of the same region, right.

\mathcal{H} , considered in terms of its “dual”, that is, the planar triangular lattice consisting of unit triangles, \mathcal{T} , drawn so that one of the families of lattice lines is vertical. In this context a matching between two neighbouring vertices of \mathcal{H} corresponds to joining a pair of unit triangles in \mathcal{T} that share precisely one edge, therefore a dimer covering of \mathcal{H} (from which a finite number of vertices may have been removed) corresponds to a tiling of the plane by unit rhombi (where a corresponding set of unit triangles have been removed). An example of a dimer covering of a subregion of \mathcal{H} and its corresponding rhombus tiling on \mathcal{T} may be found in Figure 2.

It was conjectured by Ciucu [7] in 2008 that the interaction between any set of holes, \mathcal{H} , that lie far apart within a sea of unit rhombi is asymptotically equal to

$$\prod_{h \in \mathcal{H}} \tilde{\omega}(h) \prod_{1 \leq i < j \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{2}q(h_i)q(h_j)},$$

where $\tilde{\omega}(h)$ is a constant dependent on each individual hole, $q(h)$ is the *charge*¹ of the hole h , and $d(h_i, h_j)$ is the Euclidean distance between the holes h_i and h_j indexed by \mathcal{H} . Although this conjecture remains open, the main result of this paper shows that it holds for a general family of holes that appear to have not yet been considered in any of the literature, namely triangular holes of any even side length that are *horizontally collinear* (that is, they lie on a horizontal line about which they are symmetrically distributed), where the sum of the charges of the holes is zero.² An example of such a family of holes may be found in Figure 1.

Theorem 1. *The interaction between a set \mathcal{H} of horizontally collinear triangular holes of any even side length is asymptotically*

$$\omega_H(\mathcal{H}) \sim \prod_{h \in \mathcal{H}} C_h \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{2}q(h_i)q(h_j)}$$

¹The charge of h is a statistic on the hole given by the number of right pointing unit triangles that comprise it minus the number of left pointing ones. For example a right pointing triangular hole of side length two has charge 2, whereas a left pointing hole of the same size has charge -2 .

²It shall be assumed that all sets of holes considered in this paper satisfy this condition.

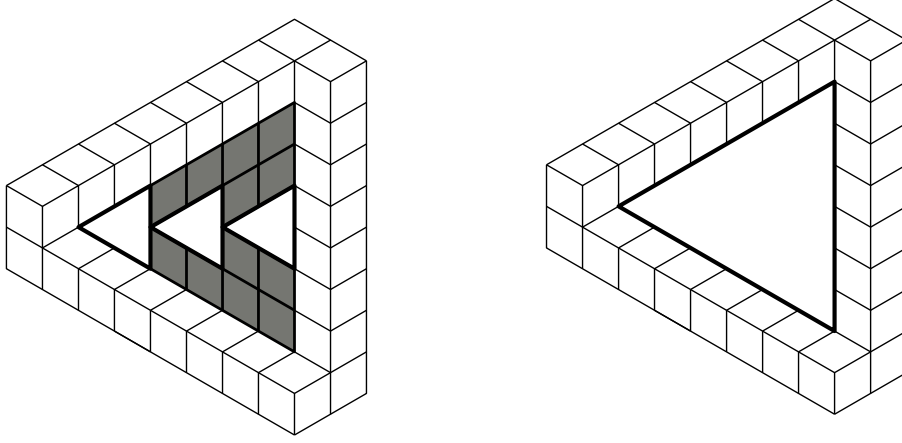


FIGURE 4. A set of contiguous triangular holes of side length two where the forced unit rhombi are coloured grey, left, and the larger induced hole of side length six, right.

as the distance between the holes h_i and h_j in \mathcal{H} becomes large, where

$$C_h = \prod_{s=0}^{\frac{1}{2}|q(h)|-1} \frac{3^{s+1/2}}{2\pi} \Gamma(s+1)^2.$$

Remark 1. The above theorem shows that the interaction between sets of interleaving, collinear triangular holes of any even side length may be approximated, up to a multiplicative constant, by taking the exponential of the negative of the electrostatic energy of the system obtained by viewing each hole as a point charge with signed magnitude given by the statistic q . A more detailed discussion of the close analogies between rhombus tilings of regions containing holes and certain electrostatic phenomena may be found in [7] and [9].

In order to prove Theorem 1 an exact formula (Theorem 3) is established in Section 3 that counts rhombus tilings of a *holey hexagon*³ centred at some origin, O , with sides of length $n, 2m, n, n, 2m, n$ (going clockwise from the southwest side), containing p -many left and $(p$ -many) right pointing horizontally collinear triangular holes of side length two lying along the horizontal line that intersects the origin. Such a region is denoted $H_{n,2m}^{L,R}$, where $R = \{r_1, \dots, r_p\}$ and $L = \{l_1, \dots, l_p\}$ are sets of unique integers that correspond to lattice distances of the midpoint of the vertical sides of the right and left pointing holes (respectively) from O . An example of a holey hexagon may be found in Figure 3.

Remark 2. A string of k -many contiguous⁴ triangular holes of side length two is equivalent to a triangular hole of side length $2k$, since dimers are forced within the “folds” of the holes (see Figure 4), thus inducing a larger hole. It is therefore sufficient to consider holey hexagons containing holes of side length two, and holes of a larger even side length shall be referred to as *induced holes*.

³This term was first coined by Propp in [20] to describe a hexagonal region that contains a set of holes in its interior.

⁴A set of horizontally collinear holes are *contiguous* if no horizontal rhombi can fit between them.

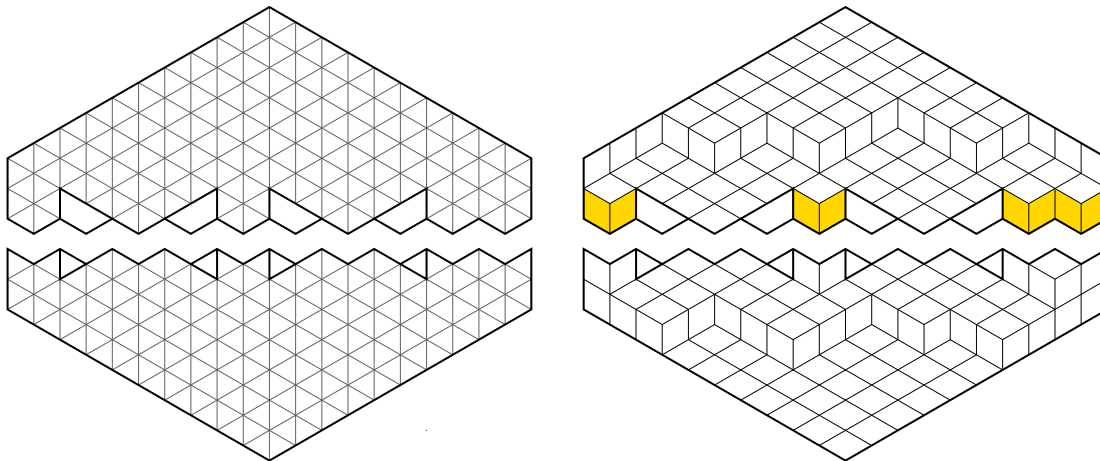


FIGURE 5. The regions $\widehat{H}_{10,4}^{\{-2,6\},\{0,-8\}}$, upper left, and $\check{H}_{10,4}^{\{-2,6\},\{0,-8\}}$, lower left, together with tilings of each region, right, where the pairs of yellow tiles have a combined weight of 2.

Theorem 3 follows from splitting $H_{n,2m}^{L,R}$ into two subregions, each obtained by cutting along the zig-zag line that proceeds just below the horizontal line intersecting the origin. The upper region is denoted $\widehat{H}_{n,2m}^{L,R}$, the lower $\check{H}_{n,2m}^{L,R}$. According to Ciucu's factorisation theorem [2],

$$M(H_{n,2m}^{L,R}) = M(\check{H}_{n,2m}^{L,R}) \cdot M_w(\widehat{H}_{n,2m}^{L,R}), \quad (2.2)$$

where $M_w(\widehat{H}_{n,2m}^{L,R})$ denotes the weighted count of tilings of $\widehat{H}_{n,2m}^{L,R}$, where every pair of unit rhombi that lie within the “folds” of the lower zig-zag boundary have a combined weight of 2 (see Figure 5).

Exact enumerative formulas (Theorem 5 and Theorem 7) that count (weighted) tilings of these subregions follow from translating tilings to families of non-intersecting lattice paths in the usual way. According to [12], enumerating such families of paths amounts to evaluating two determinants and the product of these determinant evaluations then yields Theorem 3 by way of Ciucu's factorisation result (2.2).

Enumerating tilings of $H_{n,2m}^{L,R}$ in this way also gives, for free, two enumeration formulas for certain symmetry classes of tilings of $H_{n,2m}^{L,R}$. It should be clear that tilings of $\check{H}_{n,2m}^{L,R}$ correspond to horizontally symmetric tilings of $H_{n,2m}^{L,R}$. Moreover, if each r in R is positive and satisfies $r = -l$ for some l in L , then according to Ciucu and Krattenthaler [8] the weighted count of tilings of the upper region, $M_w(\widehat{H}_{n,2m}^{L,R})$, is equal to the number of vertically symmetric tilings of $H_{n,2m}^{L,R}$, which correspond to tilings of the left half of $H_{n,2m}^{L,R}$ constrained on the right by a vertical free boundary that intersects the origin (here a boundary is considered free if unit rhombi are permitted to protrude halfway across it). Such a region shall be denoted $V_{n,2m}^L$ and an example of a tiling of such a region may be found in Figure 6.

Tilings of $V_{n,2m}^L$ correspond to tilings of the left half of the plane constrained on the right by a vertical free boundary as n and m are sent to infinity. The correlation function of holes that lie within tilings of this half plane may be interpreted as the interaction between a set of left pointing holes and a vertical free boundary (see Figure 6, right).

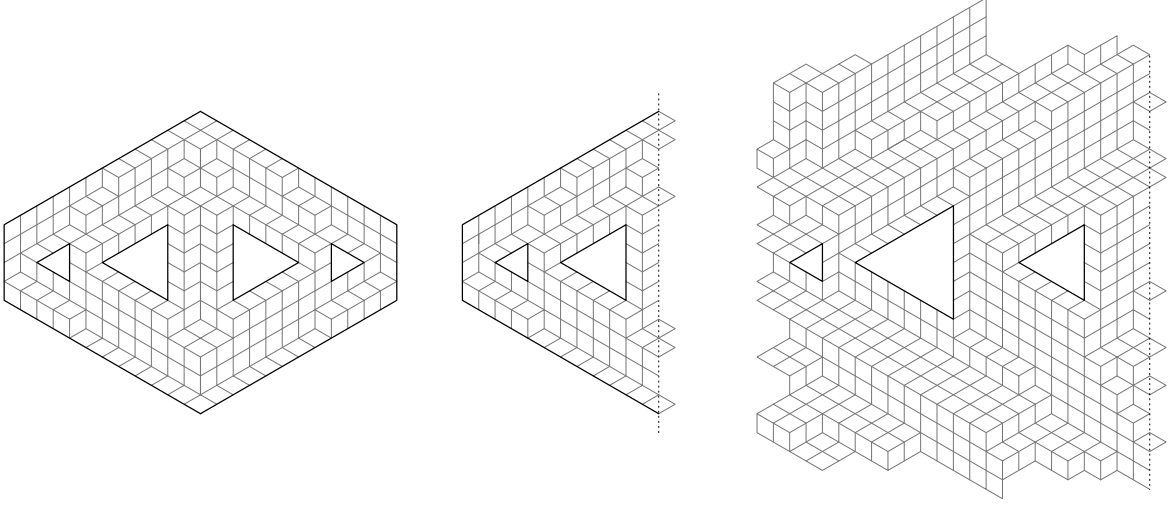


FIGURE 6. A vertically symmetric tiling of the holey hexagon $H_{12,4}^{L,R}$ where $L = \{-8, -4, -2\}$ and $R = \{2, 4, 8\}$, left, together with the corresponding tiling of $V_{12,4}^L$, centre, and a set of left pointing holes in a sea of dimers constrained on the right by a free boundary, right.

Ciucu and Krattenthaler [9] considered such an interaction for a single left pointing hole of side length two, which is a special case of the following theorem.

Theorem 2. *The interaction between a set of horizontally collinear left pointing holes of any even side length in a sea of unit rhombi and a right vertical free boundary is*

$$\omega_V(\mathcal{H}) \sim \prod_{h \in \mathcal{H}} K_h \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)},$$

where \mathcal{H} indexes both the left pointing holes and their reflections in the free boundary, and

$$K_h = \prod_{s=0}^{\frac{1}{2}|q(h)|-1} \frac{3^{s/2}\Gamma(s+1)}{\sqrt{2\pi}}.$$

Remark 3. The above result is in fact the square root of the result given in Theorem 1, and indeed this relationship is analogous to well-known and established physical laws. Consider a point charge p with signed magnitude situated near a straight line equipotential conductor. The method of images [10, Chapter 6] states that in order to calculate the electrostatic energy of the system one may replace the straight line conductor with an imaginary charge, \hat{p} , of opposite signed magnitude to that of p situated at the position specified by reflecting p through the conductor since the electric field induced by the charge(s) is the same for both arrangements (see Figure 7 for an illustration of this principle). The electrostatic energy of the system with the conductor is then one half of the electrostatic energy of the system consisting of p and its imaginary counter-charge \hat{p} . Thus the result above may be obtained, up to a multiplicative constant, by taking the exponential of the negative of half the electrostatic energy of the the system obtained by considering the set of left pointing holes described above together with their reflections through the vertical free boundary as a set of point charges with

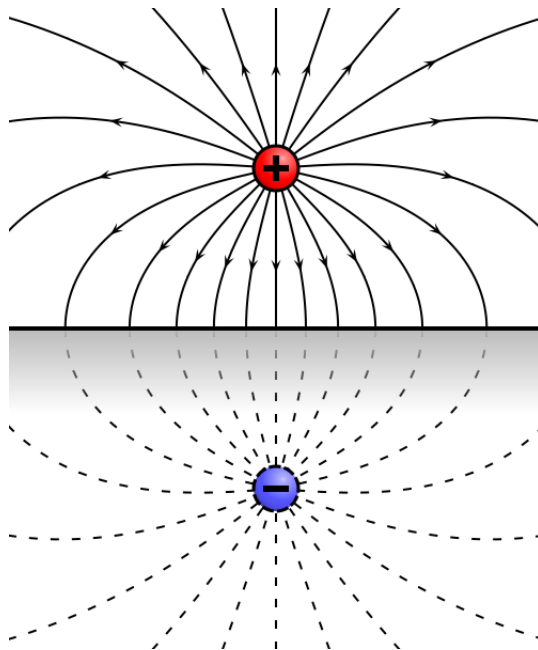


FIGURE 7. A diagram displaying the method of images, taken from [22]. The electric field induced by the positive charge, p , near the horizontal straight line conductor is the same as the electric field induced by the p and its imaginary charge \hat{p} obtained by reflecting p through the conductor.

signed magnitude given by q . One should see that this is equivalent (again up to some multiplicative constant) to taking the square root of the result given in Theorem 1.

It should be noted that Theorem 1 and Theorem 2 are specialisations of more general results that appear in Section 4, where the sides of the holey hexagon $H_{n,2m}^{L,R}$ may approach infinity at different rates (that is, $2m \sim \xi n$ for some real positive ξ). For $\xi \neq 1$ these interactions either blow up or shrink exponentially, thus the results presented above are for the special case where $\xi = 1$. While this special case agrees completely with Ciucu's tori result [5], the asymptotic analysis presented in Section 4 shows that in the planar case such an interaction depends somewhat delicately on the rate at which the “edges” of the region approach infinity. A discussion of similar findings may also be found in [9, Remark 1].

The following section establishes exact formulas that count weighted tilings of $\hat{H}_{n,2m}^{L,R}$ and $\check{H}_{n,2m}^{L,R}$. The asymptotic behaviours of these formulas are established in Section 4. Throughout both sections the following notation for generalised hypergeometric series is used:

$${}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

where $(\alpha)_\beta$ is the Pochhammer symbol, defined to be

$$(\alpha)_\beta = \begin{cases} \alpha \cdot (\alpha + 1) \cdots (\alpha + \beta - 1) & \beta > 0, \\ 1 & \beta = 0. \end{cases}$$

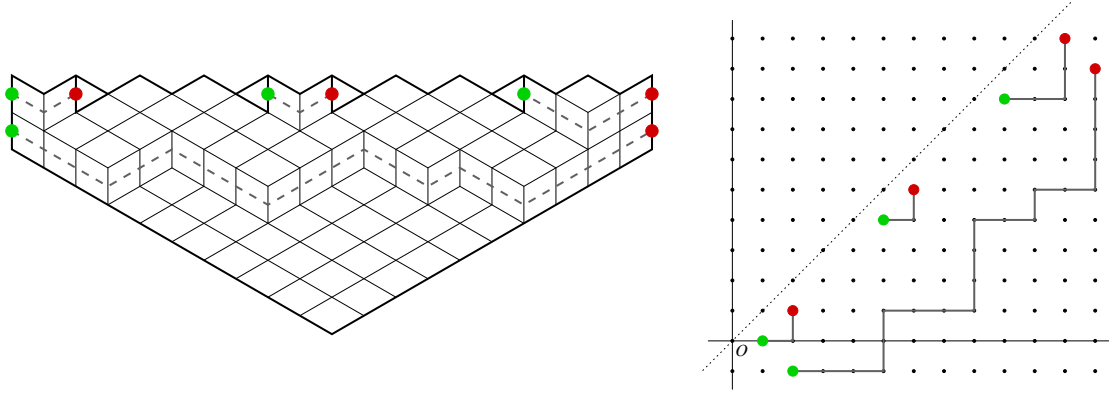


FIGURE 8. A tiling of $\check{H}_{10,4}^{\{-2,6\},\{0,-8\}}$ displaying lattice paths across unit rhombi, left, and the corresponding set of lattice paths starting at a set of green points and ending at a set of red ones, right.

3. AN EXACT FORMULA

The main goal of this section is to prove the following enumerative formula.

Theorem 3. *The number of tilings of $H_{n,2m}^{L,R}$ is*

$$\left(\prod_{i=1}^n \prod_{j=1}^{2m} \prod_{k=1}^n \frac{i+j+k-1}{i+j+k-2} \right) \cdot \det \check{E}_{R,L} \cdot \det \hat{E}_{R,L},$$

where $\check{E}_{R,L}$ and $\hat{E}_{R,L}$ are $p \times p$ matrices defined in Theorems 5 and 7 below.

Remark 4. The leftmost product in the above theorem is easily recognisable as MacMahon's much celebrated box formula [18], named so because it counts the number of unit cube representations of plane partitions that fit inside an $n \times 2m \times n$ box. This is equivalent to the number of rhombus tilings of an *un-holey hexagon*⁵ of side lengths $n, 2m, n, n, 2m, n$ (going clockwise from the southwest edge) and is denoted $H_{n,2m}$.

The above theorem follows from establishing exact formulas that enumerate the (weighted) number of tilings of the regions $\hat{H}_{n,2m}^{L,R}$ and $\check{H}_{n,2m}^{L,R}$. According to Ciucu's factorisation theorem (2.2), the product of these formulas counts the total number of tilings of $H_{n,2m}^{L,R}$. To begin, one translates tilings of these regions into families of lattice paths across dimers in the usual way⁶, which in turn correspond to families of non-intersecting lattice paths consisting of north and east steps that begin at a set of points

⁵A hexagon that contains no holes.

⁶To generate such a family, start points are set in the centre of the vertical edges of dimers that lie along the west edge of each region, along with the dimers that lie along the vertical edge of each left pointing triangular hole. One then draws a path across dimers by travelling from one side of a dimer to the opposite parallel side, thereby creating a family of non-intersecting paths that corresponds to precisely one tiling.

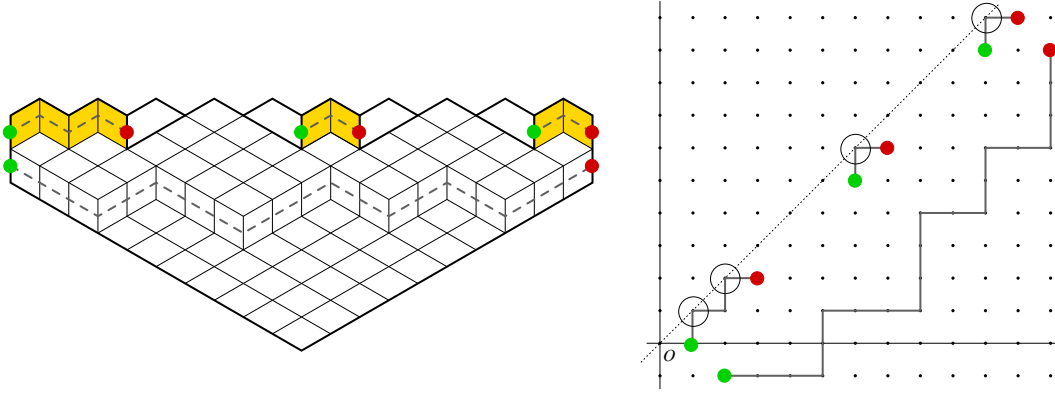


FIGURE 9. A tiling of $\widehat{H}_{10,4}^{\{-2,6\},\{0,-8\}}$ displaying lattice paths across unit rhombi, left, and the corresponding set of lattice paths starting at a set of green points and ending at a set of red ones, right, where the yellow weighted tiles on the left correspond to the circled points the touch the main diagonal on the right.

$A = (A_1, \dots, A_{m+p})$ and end at a set of points $E = (E_1, \dots, E_{m+p})$, where

$$A_i = \begin{cases} (i, 1-i) & 1 \leq i \leq m, \\ \left(\frac{n}{2} + \frac{l_{i-m}}{2} + 1, \frac{n}{2} + \frac{l_{i-m}}{2}\right) & m+1 \leq i \leq m+p, \end{cases}$$

and

$$E_j = \begin{cases} (n+j, n+1-j) & 1 \leq j \leq m, \\ \left(\frac{n}{2} + \frac{r_{j-m}}{2} + 1, \frac{n}{2} + \frac{r_{j-m}}{2}\right) & m+1 \leq j \leq m+p. \end{cases}$$

Tilings of $\check{H}_{n,2m}^{L,R}$ correspond to families of non-intersecting lattice paths that begin at A and end at E such that no path touches the main diagonal (that is, the line $y = x$), see Figure 8. The weighted count of tilings of $\widehat{H}_{n,2m}^{L,R}$ instead correspond to the weighted count of families of non-intersecting paths from A to E that do not extend above the main diagonal, and where each path P that touches the main diagonal at $T(P)$ many points has a weight of $2^{T(P)}$ (see Figure 9).

Remark 5. Suppose σ is a permutation on $m+p$ letters that maps each start point A_i to an end point $E_{\sigma(i)}$. It should be clear that due to the constraints on the families of paths described above, every family of non-intersecting paths arises from precisely one permutation which is uniquely determined by the positioning of the triangular holes relative to each other.

According to the well-known theorem of Lindström [17], Gessel, and Viennot [12], if σ is the only permutation on $m+p$ letters that gives rise to a family of non-intersecting paths that begin at A and end at E , then the number of such (weighted) paths is given by the determinant of the matrix $P = (P_{i,j})_{1 \leq i,j \leq m+p}$, where $P_{i,j}$ denotes the weighted count of the number of lattice paths from A_i to E_j . To be more precise, suppose that every vertical or horizontal unit step (edge) between a point a and a point b has weight

$e_w(a, b)$ (where the weights are elements of some commutative ring). Then the weight of a path \mathcal{P} is the product over all the unit edges that comprise it, thus if $\mathcal{P}(A_i \rightarrow E_j)$ denotes the family of paths beginning at a point A_i and ending at a point E_j then

$$P_{i,j} = \sum_{\mathcal{P} \in \mathcal{P}(A_i \rightarrow E_j)} \prod_{(a,b) \in \mathcal{P}} e_w(a, b).$$

Note that if all edge weights are 1 then $P_{i,j}$ is simply the number of paths from A_i to E_j .

Remark 6. It shall be assumed from now on that the only permutation σ that gives rise to a family of non-intersecting paths from A to E is the identity, for if not one may permute the labels of the start or end points in such a way that this holds. This corresponds to interchanging rows or columns in the matrix P , thus the number of such non-intersecting lattice paths from A to E is given by the absolute value of its determinant.

For tilings of the region $\check{H}_{n,2m}^{L,R}$, each path from A_i to E_j has a weight of 1, thus the number of non-intersecting lattice paths from A to E is given by

$$|\det(\mathcal{P}'(A_i \rightarrow E_j))_{1 \leq i,j \leq m+p}|,$$

where the function $\mathcal{P}'((a, b) \rightarrow (c, d))$ denotes the number of lattice paths that start at the point (a, b) and end at the point (c, d) and never touch the main diagonal. A straightforward argument shows that the number of such paths is given by

$$\mathcal{P}'((a, b) \rightarrow (c, d)) = \mathcal{P}((a, b) \rightarrow (c, d)) - \mathcal{P}((a, b) \rightarrow (d, c)),$$

where $\mathcal{P}((a, b) \rightarrow (c, d))$ denotes the function that counts the number of ordinary lattice paths beginning at the point (a, b) and ending at the point (c, d) . Since this function \mathcal{P} is given by a binomial coefficient it follows that

$$M(\check{H}_{n,2m}^{L,R}) = |\det \check{Q}|,$$

where the matrix $\check{Q} = (\check{Q}_{i,j})_{1 \leq i,j \leq m+p}$ has (i, j) -entries given by

$$\check{Q}_{i,j} = \begin{cases} \binom{2n}{n+j-i} - \binom{2n}{n+1-i-j} & 1 \leq i, j \leq m, \\ \frac{2i-1}{n+r_{j-m}+1} \binom{n+r_{j-m}+1}{n/2+r_{j-m}/2+1-i} & i \in \{1, \dots, m\}, j \in \{m+1, \dots, m+p\}, \\ \frac{2j-1}{n-l_{i-m}+1} \binom{n-l_{i-m}+1}{n/2-l_{i-m}/2-1+j} & i \in \{m+1, \dots, m+p\}, j \in \{1, \dots, m\}, \\ \frac{1}{r_{j-m}-l_{i-m}+1} \binom{r_{j-m}-l_{i-m}+1}{r_{j-m}/2-l_{i-m}/2} & 1 \leq i, j \leq m+p. \end{cases}$$

Remark 7. In the above definitions (and indeed throughout this article) the binomial coefficient shall be interpreted in the “natural” way, that is, for integers n and k ,

$$\binom{n}{k} = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4. *The matrix \check{Q} defined above has the following LU-decomposition*

$$\check{Q} = L \cdot U,$$

where $L = (L_{i,j})_{1 \leq i,j \leq m+p}$ has (i,j) -entries given by

$$L_{i,j} = \begin{cases} A_n(i,j) & 1 \leq j < i \leq m, \\ B_n(i,j) & m+1 \leq i \leq m+p, 1 \leq j \leq m, \\ E_{n,m}(i,j) & m+1 \leq j < i \leq m+p, \\ 1 & i = j, 1 \leq j \leq m+p, \\ 0 & \text{otherwise,} \end{cases}$$

and $U = (U_{i,j})_{1 \leq i,j \leq m+1}$ is given by

$$U_{i,j} = \begin{cases} C_n(i,j) & 1 \leq i \leq j \leq m, \\ D_n(i,j) & 1 \leq i \leq m, m+1 \leq j \leq m+p, \\ F_{n,m}(i,j) & m+1 \leq j \leq i \leq m+p, \\ 0 & \text{otherwise,} \end{cases}$$

such that

$$\begin{aligned} A_n(i,j) &= \frac{\Gamma(2i)\Gamma(n+1)\Gamma(i+j-1)\Gamma(2j+n)}{\Gamma(2i-1)\Gamma(2j)\Gamma(i-j+1)\Gamma(j-i+n+1)\Gamma(i+j+n)}, \\ B_n(i,j) &= \frac{(-1)^{j+1}\Gamma(j+n-1)\Gamma(2j+n)\Gamma(n-l_{i-m}+1)\Gamma(j+\frac{l_{i-m}}{2}+\frac{n}{2}-1)}{2\Gamma(j)\Gamma(2j+2n-2)\Gamma(\frac{n}{2}-\frac{l_{i-m}}{2}+1)\Gamma(\frac{l_{i-m}}{2}+\frac{n}{2})\Gamma(j-\frac{l_{i-m}}{2}+\frac{n}{2}+1)}, \\ C_n(i,j) &= \frac{\Gamma(2j)\Gamma(n+1)\Gamma(i+j-1)\Gamma(2i+2n-1)}{\Gamma(2j-1)\Gamma(j-i+1)\Gamma(2i+n-1)\Gamma(i-j+n+1)\Gamma(i+j+n)}, \\ D_n(i,j) &= \frac{(-1)^{i+1}\Gamma(2i+1)\Gamma(i+n)\Gamma(n+r_{j-m}+1)\Gamma(i+\frac{n}{2}-\frac{r_{j-m}}{2}-1)}{2\Gamma(2i+n-1)\Gamma(i+1)\Gamma(\frac{n}{2}-\frac{r_{j-m}}{2})\Gamma(\frac{n}{2}+\frac{r_{j-m}}{2}+1)\Gamma(i+\frac{n}{2}+\frac{r_{j-m}}{2}+1)}, \end{aligned}$$

and $E_{n,m}(i,j)$ and $F_{n,m}(i,j)$ are functions satisfying

$$\check{Q}_{i,j} = \sum_{s=1}^m B_n(i,s)D_n(s,j) + \sum_{s=m+1}^{\min(i,j)} E_{n,m}(i,s)F_{n,m}(s,j)$$

for $m+1 \leq i, j \leq m+p$.

Proof. The proof of the above proposition is very similar to that of Theorem 5.5 in [13] in that it relies on showing that the following identities hold:

- (i) $\sum_{s=1}^{\min(i,j)} A_n(i,s)C_n(s,j) = \binom{2n}{n+j-i} - \binom{2n}{n+i-j}$;
- (ii) $\sum_{s=1}^i A_n(i,s)D_n(s,j) = \frac{2i-1}{n+r_{j-m}+1} \binom{n+r_{j-m}+1}{n/2+r_{j-m}/2+1-i}$;
- (iii) $\sum_{s=1}^j B_n(i,s)C_n(s,j) = \frac{2j-1}{n-l_{i-m}+1} \binom{n-l_{i-m}+1}{n/2-l_{i-m}/2+1-j}$.

Case (i) follows immediately from the observation that $A_n(i,j)$ and $C_n(i,j)$ defined above are equal to $A'_n(i,j)$ and $C'_n(i,j)$ (respectively) in Theorem 5.5 of the aforementioned article. Similarly, by replacing k with $-r_{j-m}$ in the proof of case (ii) of that same theorem one sees that case (ii) above also holds.

The proof of the third case is in much the same vein, since the left hand side satisfies the following recurrence

$$(2j+1)(2j+l_{i-m}-n-2) \sum_{s=1}^j B_n(i,s)C_n(s,j) \\ + (2j-1)(2j+l_{i-m}-n-2) \sum_{s=1}^{j+1} B_n(i,s)C_n(s,j+1) = 0,$$

and one may easily check that the above equation holds when the sums are replaced with the corresponding expression from the right hand side of case (iii). Verifying this expression holds for initial conditions completes the proof. \square

Remark 8. The third case follows directly from the observation that $B_n(i,s)C_n(s,j)$ is equal to $A_n(j,s)D_n(s,i)$ with r_{j-m} replaced by $-l_{i-m}$. The proof is presented above in such a way as to give a flavour of the overall approach required to prove many of the results in [13], some of which are indeed specialisations of the results presented here.

Theorem 5. *The number of rhombus tilings of $\check{H}_{n,2m}^{L,R}$ is*

$$\left(\binom{n+m-1}{n-1} \prod_{i=1}^{n-2} \prod_{j=i}^{n-2} \frac{2m+i+j+1}{i+j+1} \right) \cdot |\det \check{E}_{R,L}|,$$

where $\check{E}_{R,L}$ is the $p \times p$ matrix with (i,j) -entries given by

$$\check{e}_{i,j} = {}_4F_3 \left[\begin{matrix} \frac{r}{2} - \frac{n}{2} + 1, 1, \frac{r}{2} - \frac{l}{2} + 2, \frac{n}{2} + \frac{r}{2} + \frac{1}{2} \\ m + \frac{n}{2} + \frac{r}{2} + 2, \frac{r}{2} - m - \frac{n}{2} + 2, \frac{r}{2} - \frac{l}{2} + \frac{3}{2} \end{matrix}; 1 \right] \\ \times \frac{\Gamma(m+n+1)\Gamma(\frac{n}{2} + \frac{r_j}{2} + \frac{1}{2})\Gamma(\frac{l_i}{2} + m + \frac{n}{2})\Gamma(m + \frac{n}{2} - \frac{r_j}{2} - 1)}{\Gamma(\frac{n}{2} - \frac{r_j}{2})\Gamma(m - \frac{l_i}{2} + \frac{n}{2} + 1)\Gamma(m + \frac{n}{2} + \frac{r_j}{2} + 2)} \\ \times \frac{2^{r_j-l_i+2}\Gamma(m + \frac{3}{2})\Gamma(\frac{n}{2} - \frac{l_i}{2} + \frac{1}{2})}{\pi(r_j - l_i + 1)\Gamma(m)\Gamma(\frac{l_i}{2} + \frac{n}{2})\Gamma(m+n - \frac{1}{2})}$$

if $r_j > l_i$ and

$$\check{e}_{i,j} = - \frac{2^{-l_i+r_j+2}\Gamma(m + \frac{3}{2})\Gamma(\frac{1}{2}(-l_i+n+1))\Gamma(m+n+1)\Gamma(\frac{1}{2}(n+r_j+1))}{3\pi\Gamma(m)\Gamma(\frac{1}{2}(-l_i+n+4))\Gamma(m+n - \frac{1}{2})\Gamma(\frac{1}{2}(n+r_j+4))} \\ \times {}_4F_3 \left[\begin{matrix} 2 - \frac{l_i}{2} + \frac{r_j}{2}, \frac{3}{2}, m+n+1, 1-m \\ \frac{n}{2} + 2 - \frac{l_i}{2}, \frac{n}{2} + \frac{r_j}{2} + 2, \frac{5}{2} \end{matrix}; 1 \right]$$

otherwise.

Proof. The LU -decomposition in Proposition 4 is unique since all entries on the diagonal of L are 1. Thus the determinant of the matrix \check{Q} is

$$\prod_{s=1}^{m+p} U_{s,s} = \left(\prod_{s=1}^m C_n(s,s) \right) \cdot \prod_{t=m+1}^{m+p} U_{t,t}.$$

The product of $C_n(s, s)$ over s above is equal to the well-known formula due to Proctor [19] that counts transpose complementary plane partitions in an $n \times 2m \times n$ box (equivalently, horizontally symmetric tilings of $H_{n,2m}$),

$$\binom{n+m-1}{n-1} \prod_{i=1}^{n-2} \prod_{j=i}^{n-2} \frac{2m+i+j+1}{i+j+1}.$$

Although the entries of $U_{t,t}$ have not been explicitly established for $1 \leq t \leq m+p$, it should be clear that their product is equal to the determinant of a certain $p \times p$ matrix $\check{E}_{R,L} = (\check{e}_{i,j})_{1 \leq i,j \leq p}$ with entries given by

$$\begin{aligned} \check{e}_{i,j} &= \sum_{s=m+1}^{\min(i,j)} E_{n,m}(m+i, s) F_{n,m}(s, m+j) \\ &= \check{Q}_{m+i, m+j} - \sum_{s=1}^m B_n(m+i, s) D_n(s, m+j). \end{aligned}$$

Suppose first that $r_j > l_i$. Then the entry $\check{e}_{i,j}$ may be re-written as

$$\begin{aligned} \frac{1}{r_{j-m} - l_{i-m} + 1} \binom{r_{j-m} - l_{i-m} + 1}{r_{j-m}/2 - l_{i-m}/2} - \sum_{s=0}^{\infty} B_n(m+i, s+1) D_n(s+1, m+j) \\ + \sum_{t=m}^{\infty} B_n(m+i, t+1) D_n(t+1, m+j). \end{aligned}$$

The above sum over s may be expressed as the following hypergeometric series

$$\begin{aligned} \frac{\Gamma(n)\Gamma(n+2)\Gamma(n-l+1)\Gamma(n+r+1)}{2\Gamma(2n)\Gamma(\frac{n}{2} - \frac{l}{2} + 1)\Gamma(\frac{n}{2} - \frac{l}{2} + 2)\Gamma(\frac{n}{2} + \frac{r}{2} + 1)\Gamma(\frac{n}{2} + \frac{r}{2} + 2)} \\ \times {}_5F_4 \left[\begin{matrix} n+1, \frac{n}{2} + \frac{3}{2}, \frac{3}{2}, \frac{l}{2} + \frac{n}{2}, \frac{n}{2} - \frac{r}{2} \\ \frac{n}{2} + \frac{1}{2}, n + \frac{1}{2}, \frac{n}{2} - \frac{l}{2} + 2, \frac{n}{2} + \frac{r}{2} + 2 \end{matrix}; 1 \right], \quad (3.1) \end{aligned}$$

(note the abuse of notation: r_j and l_i have been replaced with r and l respectively). According to Slater [21, Appendix III.12] this hypergeometric series satisfies the following summation formula

$$\begin{aligned} {}_5F_4 \left[\begin{matrix} a, \frac{a}{2} + 1, b, c, d \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1 \end{matrix}; 1 \right] \\ = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-b-c-d+1)}{\Gamma(a+1)\Gamma(a-b-c+1)\Gamma(a-b-d+1)\Gamma(a-c-d+1)}. \end{aligned}$$

Applying this formula to (3.1) yields

$$\frac{\Gamma(n)\Gamma(n - \frac{1}{2} + 1)\Gamma(n-l+1)\Gamma(\frac{r}{2} - \frac{l}{2} + \frac{1}{2})\Gamma(n+r+1)}{2\Gamma(2n)\Gamma(\frac{n}{2} - \frac{l}{2} + \frac{1}{2})\Gamma(\frac{n}{2} - \frac{l}{2} + 1)\Gamma(\frac{r}{2} - \frac{l}{2} + 2)\Gamma(\frac{n}{2} + \frac{r}{2} + \frac{1}{2})\Gamma(\frac{n}{2} + \frac{r}{2} + 1)},$$

which may easily be shown to equal

$$\frac{1}{r-l+1} \binom{r-l+1}{r/2 - l/2},$$

hence for $r_j > l_i$ the entries $\check{e}_{i,j}$ are given by

$$\sum_{t=0}^{\infty} B_n(m+i, t+m+1) D_n(t+m+1, m+j).$$

Prolonging the aforementioned abuse of notation, the above sum may be written as the following hypergeometric series

$$\begin{aligned} {}_6F_5 \left[\begin{matrix} m + \frac{n}{2} + \frac{3}{2}, m + \frac{3}{2}, \frac{l}{2} + m + \frac{n}{2}, m + n + 1, m + \frac{n}{2} - \frac{r}{2}, 1 \\ m + \frac{n}{2} + \frac{1}{2}, m + n + \frac{1}{2}, m - \frac{l}{2} + \frac{n}{2} + 2, m + 1, m + \frac{n}{2} + \frac{r}{2} + 2 \end{matrix}; 1 \right] \\ \times \frac{\Gamma(2m+3)\Gamma(n-l+1)\Gamma(m+n)\Gamma(m+n+1)\Gamma(2m+n+2)}{4\Gamma(m+1)\Gamma(m+2)\Gamma(\frac{n}{2} - \frac{l}{2} + 1)\Gamma(\frac{l}{2} + \frac{n}{2})\Gamma(2m+n+1)\Gamma(2m+2n)} \\ \times \frac{\Gamma(n+r+1)\Gamma(\frac{l}{2} + m + \frac{n}{2})\Gamma(m + \frac{n}{2} - \frac{r}{2})}{\Gamma(\frac{n}{2} - \frac{r}{2})\Gamma(\frac{n}{2} + \frac{r}{2} + 1)\Gamma(m - \frac{l}{2} + \frac{n}{2} + 2)\Gamma(m + \frac{n}{2} + \frac{r}{2} + 2)}. \end{aligned} \quad (3.2)$$

The ${}_6F_5$ hypergeometric series in the above expression may be written as the following limit of a ${}_7F_6$ series

$$\lim_{\epsilon \rightarrow 0} {}_7F_6 \left[\begin{matrix} 2m+n+1+\epsilon, V \\ W \end{matrix}; 1 \right], \quad (3.3)$$

where V and W are the lists

$$(m + \frac{n}{2} + \frac{3}{2} + \frac{\epsilon}{2}, m + \frac{3}{2} + \frac{\epsilon}{2}, \frac{l}{2} + m + \frac{n}{2} + \frac{\epsilon}{2}, 1 + \frac{\epsilon}{2}, m + n + 1 + \frac{\epsilon}{2}, m + \frac{n}{2} - \frac{r}{2} + \frac{\epsilon}{2})$$

and

$$(m + \frac{n}{2} + \frac{1}{2} + \frac{\epsilon}{2}, m + n + \frac{1}{2} + \frac{\epsilon}{2}, m - \frac{l}{2} + \frac{n}{2} + 2 + \frac{\epsilon}{2}, 2m + n + 1 + \frac{\epsilon}{2}, m + 1 + \frac{\epsilon}{2}, m + \frac{n}{2} + \frac{r}{2} + 2 + \frac{\epsilon}{2})$$

respectively. Such a series satisfies the following transformation formula

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} a, \frac{a}{2} + 1, b, c, d, e, a - e + n + 1 \\ \frac{a}{2}, a - b + 1, a - c + 1, a - d + 1, a - e + 1, e - n \end{matrix}; 1 \right] \\ = \frac{\Gamma(a-d+1)\Gamma(a-c+1)\Gamma(a-b+1)\Gamma(a-b-c-d+1)}{\Gamma(a-c-d+1)\Gamma(a-b-d+1)\Gamma(a-b-c+1)\Gamma(a+1)} \\ \times {}_4F_3 \left[\begin{matrix} b, c, d, -n \\ a - e + 1, -a + b + c + d, e - n \end{matrix}; 1 \right], \end{aligned}$$

which may also be found in Slater's book [21, (4.3.6.4) reversed]. Applying this transformation to (3.3), permuting the elements in the hypergeometric series and letting ϵ tend to zero one obtains

$$\frac{(2m+2n-1)(2m-l+n+2)}{2(n-l-1)(2m+n+1)} {}_4F_3 \left[\begin{matrix} 1, \frac{l}{2} + m + \frac{n}{2}, \frac{r}{2} - \frac{n}{2} + 1, m + \frac{3}{2} \\ m + 1, \frac{l}{2} - \frac{n}{2} + \frac{3}{2}, m + \frac{n}{2} + \frac{r}{2} + 2 \end{matrix}; 1 \right]. \quad (3.4)$$

One final transformation formula (see Slater [21, (4.3.5.1)]),

$$\begin{aligned} {}_4F_3 \left[\begin{matrix} a, b, c, -n \\ e, f, a+b+c-e-f-n+1 \end{matrix}; 1 \right] \\ = {}_4F_3 \left[\begin{matrix} -n, a, a+c-e-f-n+1, a+b-e-f-n+1 \\ a+b+c-e-f-n+1, a-e-n+1, a-f-n+1 \end{matrix}; 1 \right] \\ \times \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}, \end{aligned}$$

applied to the ${}_4F_3$ series in (3.4) gives

$$\begin{aligned} \frac{m(2m+2n-1)(2m+n-l+2)}{(r-l+1)(2m+n+1)(2m+n-r-2)} \\ \times {}_4F_3 \left[\begin{matrix} \frac{r}{2} - \frac{n}{2} + 1, 1, \frac{r}{2} - \frac{l}{2} + 2, \frac{n}{2} + \frac{r}{2} + \frac{1}{2} \\ m + \frac{n}{2} + \frac{r}{2} + 2, \frac{r}{2} - m - \frac{n}{2} + 2, \frac{r}{2} - \frac{l}{2} + \frac{3}{2} \end{matrix}; 1 \right] \end{aligned}$$

as an equivalent expression for the ${}_6F_5$ series stated above.

Suppose now that $r_j < l_i$. Then according to Remark 7

$$\check{e}_{i,j} = - \sum_{s=1}^m B_n(m+i, s) D_n(s, m+j),$$

the right hand side of which may be expressed as the following limit of a hypergeometric series

$$\frac{\Gamma(n)\Gamma(n+2)\Gamma(n-l_i+1)\Gamma(n+r_j+1)}{2\Gamma(2n)\Gamma(\frac{n}{2}-\frac{l_i}{2}+1)\Gamma(\frac{n}{2}-\frac{l_i}{2}+2)\Gamma(\frac{n}{2}+\frac{r_j}{2}+1)\Gamma(\frac{n}{2}+\frac{r_j}{2}+2)} \lim_{\epsilon \rightarrow 0} \left({}_7F_6 \left[\begin{matrix} V \\ W \end{matrix}; 1 \right] \right), \quad (3.5)$$

where

$$V = (\epsilon + n + 1, \frac{\epsilon}{2} + \frac{n}{2} + \frac{3}{2}, \frac{\epsilon}{2} + \frac{l_i}{2} + \frac{n}{2}, \frac{\epsilon}{2} + \frac{n}{2} - \frac{r_j}{2}, \frac{\epsilon}{2} + \frac{3}{2}, m + n + 1, 1 - m)$$

and

$$W = (\frac{\epsilon}{2} + \frac{n}{2} + \frac{1}{2}, \frac{\epsilon}{2} - \frac{l_i}{2} + \frac{n}{2} + 2, \frac{\epsilon}{2} + \frac{n}{2} + \frac{r_j}{2} + 2, \frac{\epsilon}{2} + n + \frac{1}{2}, \epsilon - m + 1, \epsilon + m + n + 1).$$

By applying the following transformation formula,

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} a, \frac{a}{2} + 1, b, c, d, e, -n \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1, a+n+1 \end{matrix}; 1 \right] \\ = \frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n} {}_4F_3 \left[\begin{matrix} a-b-c+1, d, e, -n \\ a-b+1, a-c+1, -a+d+e-n \end{matrix}; 1 \right] \end{aligned} \quad (3.6)$$

(see [21, (2.4.1.1), reversed]) to (3.5) and letting ϵ tend to zero, one obtains

$$\begin{aligned} \check{e}_{i,j} = - \frac{2^{r_j-l_i+2}\Gamma(m+\frac{3}{2})\Gamma(\frac{n}{2}-\frac{l_i}{2}+\frac{1}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2}+\frac{r_j}{2}+\frac{1}{2})}{3\pi\Gamma(m)\Gamma(\frac{n}{2}-\frac{l_i}{2}+2)\Gamma(m+n-\frac{1}{2})\Gamma(\frac{n}{2}+\frac{r_j}{2}+2)} \\ \times {}_4F_3 \left[\begin{matrix} 2 - \frac{l_i}{2} + \frac{r_j}{2}, \frac{3}{2}, m+n+1, 1-m \\ \frac{n}{2} + 2 - \frac{l_i}{2}, \frac{n}{2} + \frac{r_j}{2} + 2, \frac{5}{2} \end{matrix}; 1 \right] \end{aligned}$$

thus determining $\check{e}_{i,j}$ completely. \square

What remains is to determine a formula that counts the weighted tilings of $\widehat{H}_{n,2m}^{L,R}$. According to Lindström [17], Gessel and Viennot [12], the number of families of weighted paths that count such tilings is given by

$$|\det(\mathcal{P}''(A_i \rightarrow E_j))_{1 \leq i,j \leq m+p}|,$$

where $\mathcal{P}''((a,b) \rightarrow (c,d))$ denotes the number of paths from the point (a,b) to the point (c,d) that do not extend above the main diagonal, with the added condition that each path P that touches the main diagonal at $T(P)$ many points has a weight of $2^{T(P)}$. It is straightforward to show (an argument may be found in [8]) that

$$\mathcal{P}''((a,b) \rightarrow (c,d)) = \mathcal{P}((a,b) \rightarrow (c,d)) + \mathcal{P}((a,b) \rightarrow (d,c)),$$

whence

$$M_w(\widehat{H}_{n,2m}^{L,R}) = |\det \widehat{Q}|,$$

where $\widehat{Q} = (\widehat{Q}_{i,j})_{1 \leq i,j \leq m+p}$ has (i,j) -entries given by

$$\widehat{Q}_{i,j} = \begin{cases} \binom{2n}{n+j-i} + \binom{2n}{n+1-i-j} & 1 \leq i, j \leq m, \\ \binom{n+r_{j-m}+1}{n/2+r_{j-m}/2+1-i} & i \in \{1, \dots, m\}, j \in \{m+1, \dots, m+p\}, \\ \binom{n-l_{i-m}+1}{n/2-l_{i-m}/2+1-j} & i \in \{m+1, \dots, m+p\}, j \in \{1, \dots, m\}, \\ \binom{r_{j-m}-l_{i-m}+1}{r_{j-m}/2-l_{i-m}/2} & 1 \leq i, j \leq m+p. \end{cases}$$

Remark 9. As mentioned in the previous section, if the holes determined by the sets R and L are distributed symmetrically with respect to the horizontal *and* vertical symmetry axes of $H_{n,2m}^{L,R}$ (that is, all r in R are positive and satisfy $r = -l$ for some l in L), then according to [8] the number of tilings of $\widehat{H}_{n,2m}^{L,R}$ is equal to the number of tilings of $V_{n,2m}^L$ (that is, the number of vertically symmetric tilings of $H_{n,2m}^{L,R}$). This holds in particular for $L = R = \emptyset$ (in other words, vertically symmetric tilings of the un-hole hexagon $H_{n,2m}$). The proof of this striking result relies on applying elementary row and column operations to certain matrices- it would appear that a combinatorial interpretation of the relationship between these sets of tilings has yet to be realised.

Proposition 6. *The matrix \widehat{Q} defined above has LU-decomposition*

$$\widehat{Q} = L' \cdot U',$$

where $L' = (L'_{i,j})_{1 \leq i,j \leq m+p}$ is given by

$$L'_{i,j} = \begin{cases} A'_n(i,j) & 1 \leq j < i \leq m, \\ B'_n(i,j) & m+1 \leq i \leq m+p, 1 \leq j \leq m, \\ E'_{n,m}(i,j) & m+1 \leq j < i \leq m+p, \\ 1 & i = j, 1 \leq j \leq m+p, \\ 0 & \text{otherwise,} \end{cases}$$

and $U' = (U'_{i,j})_{1 \leq i,j \leq m+p}$ is given by

$$U'_{i,j} = \begin{cases} C'_n(i,j) & 1 \leq i \leq j \leq m, \\ D'_n(i,j) & 1 \leq i \leq m, m+1 \leq j \leq m+p, \\ F'_{n,m}(i,j) & m+1 \leq i \leq j \leq m+p, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned}
A'_n(i, j) &= \frac{\Gamma(n+1)\Gamma(i+j-1)\Gamma(2j+n)}{\Gamma(2j-1)\Gamma(i-j+1)\Gamma(j-i+n+1)\Gamma(i+j+n)}, \\
B'_n(i, j) &= \frac{(-1)^{j+1}\Gamma(j+n)\Gamma(2j+n)\Gamma(n-l_{i-m}+2)\Gamma(j+\frac{l_{i-m}}{2}+\frac{n}{2}-1)}{\Gamma(j)\Gamma(2j+2n)\Gamma(\frac{n}{2}-\frac{l_{i-m}}{2}+1)\Gamma(\frac{l_{i-m}}{2}+\frac{n}{2})\Gamma(j-\frac{l_{i-m}}{2}+\frac{n}{2}+1)}, \\
C'_n(i, j) &= \frac{\Gamma(n+1)\Gamma(i+j-1)\Gamma(2i+2n)}{\Gamma(j-i+1)\Gamma(2i+n-1)\Gamma(i-j+n+1)\Gamma(i+j+n)}, \\
D'_n(i, j) &= \frac{(-1)^{i+1}\Gamma(2i-1)\Gamma(i+n)\Gamma(n+r_{j-m}+2)\Gamma(i+\frac{n}{2}-\frac{r_{j-m}}{2}-1)}{\Gamma(i)\Gamma(2i+n-1)\Gamma(\frac{n}{2}-\frac{r_{j-m}}{2})\Gamma(\frac{n}{2}+\frac{r_{j-m}}{2}+1)\Gamma(i+\frac{n}{2}+\frac{r_{j-m}}{2}+1)},
\end{aligned}$$

and $E'_{n,m}(i, j)$ and $F'_{n,m}(i, j)$ are functions satisfying

$$\widehat{Q}_{i,j} = \sum_{s=1}^m B'_n(i, s) D'_n(s, j) + \sum_{s=m+1}^{\min(i,j)} E'_{n,m}(i, s) F'_{n,m}(s, j).$$

Proof. Once again, the proof of the above theorem may be reduced to proving three identities, chiefly:

- (i) $\sum_{s=1}^{\min(i,j)} A'_n(i, s) C'_n(s, j) = \binom{2n}{n+j-i} + \binom{2n}{n+1-i-j}$;
- (ii) $\sum_{s=1}^j A'_n(i, s) D'_n(s, j) = \binom{n-l_{i-m}+1}{n/2-l_{i-m}/2+1-j}$;
- (iii) $\sum_{s=1}^i B'_n(i, s) C'_n(s, j) = \binom{n+r_{j-m}+1}{n/2+r_{j-m}/2+1-i}$.

Proofs of these identities may also be found in [13, Theorem 5.2 and Lemma 5.3], since $A'_n(i, j)$, $B'_n(i, j)$, $C'_n(i, j)$, $D'_n(i, j)$ above are equal to $A_n(i, j)$, $B_{n,l_{i-m}}(j)$, $C_n(i, j)$, $E_{n,-r_{j-m}}(i)$ (respectively) from that same article. \square

Theorem 7. *The weighted count of rhombus tilings of $\widehat{H}_{n,2m}^{L,R}$ is given by*

$$\left(\prod_{i=1}^n \frac{2i+2m-1}{2i-1} \prod_{1 \leq i < j \leq n} \frac{i+j+2m-1}{i+j-1} \right) \cdot |\det \widehat{E}_{R,L}|,$$

where $\widehat{E}_{R,L}$ is the $p \times p$ matrix with (i, j) -entries given by

$$\begin{aligned}
\widehat{e}_{i,j} &= {}_4F_3 \left[\begin{matrix} \frac{r_j}{2} - \frac{n}{2} + 1, 1, \frac{r_j}{2} - \frac{l_i}{2} + 2, \frac{n}{2} + \frac{r_j}{2} + \frac{3}{2} \\ m + \frac{n}{2} + \frac{r_j}{2} + 2, \frac{r_j}{2} - m - \frac{n}{2} + 2, \frac{r_j}{2} - \frac{l_i}{2} + \frac{5}{2} \end{matrix}; 1 \right] \\
&\times \frac{\Gamma(m+n+1)\Gamma(\frac{n}{2}+\frac{r_j}{2}+\frac{3}{2})\Gamma(\frac{l_i}{2}+\frac{m}{2}+\frac{n}{2}+1)\Gamma(m+\frac{n}{2}-\frac{r_j}{2}-1)}{\Gamma(\frac{n}{2}-\frac{r_j}{2})\Gamma(\frac{n}{2}-\frac{l_i}{2}+m+1)\Gamma(\frac{n}{2}+m+\frac{r_j}{2}+2)} \\
&\times \frac{2^{-l+r+2}\Gamma(m+\frac{1}{2})\Gamma(\frac{n}{2}-\frac{l_i}{2}+\frac{3}{2})}{\pi(r-l+3)\Gamma(m)\Gamma(\frac{l_i}{2}+\frac{n}{2})\Gamma(m+n+\frac{1}{2})}
\end{aligned}$$

for $r_j < l_i$ and

$$\begin{aligned}
\widehat{e}_{i,j} &= -\frac{2^{r_j-l_i+2}\Gamma(m+\frac{1}{2})\Gamma(\frac{n}{2}-\frac{l_i}{2}+\frac{3}{2})\Gamma(m+n+1)\Gamma(\frac{n}{2}+\frac{r_j}{2}+\frac{3}{2})}{\pi\Gamma(m)\Gamma(\frac{n}{2}-\frac{l_i}{2}+2)\Gamma(m+n+\frac{1}{2})\Gamma(\frac{n}{2}+\frac{r_j}{2}+2)} \\
&\times {}_4F_3 \left[\begin{matrix} 2 + \frac{r_j}{2} - \frac{l_i}{2}, \frac{1}{2}, m+n+1, 1-m \\ \frac{n}{2} - \frac{l_i}{2} + 2, \frac{n}{2} + \frac{r_j}{2} + 2, \frac{3}{2} \end{matrix}; 1 \right]
\end{aligned}$$

otherwise.

Proof. The LU -decomposition of \widehat{Q} is unique, thus its determinant is

$$\prod_{s=1}^{m+p} U'_{s,s} = \left(\prod_{s=1}^m C'_n(s, s) \right) \cdot \prod_{t=m+1}^{m+p} U'_{t,t}.$$

The product of $C'_n(s, s)$ over s is in fact the number of tilings of the un-hole hexagon $H_{n,2m}$ that are vertically symmetric (see Remark 9). A formula for such tilings was conjectured by MacMahon [18] to be

$$\prod_{i=1}^n \frac{2i + 2m - 1}{2i - 1} \prod_{1 \leq i < j \leq n} \frac{i + j + 2m - 1}{i + j - 1},$$

which was later proved independently by both Andrews [1] and Gordon [14] (although Gordon's proof was published much later).

By the same argument that appears in the proof of Theorem 5,

$$\prod_{t=m+1}^{m+p} U'_{t,t} = \det \widehat{E}_{R,L},$$

where $\widehat{E}_{R,L}$ is a $p \times p$ matrix with (i, j) -entries given by

$$\hat{e}_{i,j} = \widehat{Q}_{m+i,m+j} - \sum_{s=1}^m B'_n(m+i, s) D'_n(s, m+j).$$

Precisely the same transformation and summation formulas from the proof of Theorem 5 may be applied to the right hand side of the above expression in order to obtain $\hat{e}_{i,j}$ as stated in the theorem. \square

Having established Theorem 5 and Theorem 7, Theorem 3 follows immediately from inserting the above expressions for $M(\check{H}_{n,2m}^{L,R})$ and $M_w(\widehat{H}_{n,2m}^{L,R})$ into Ciucu's factorisation theorem (2.2). Note that the absolute value of the determinant is no longer taken in Theorem 3 since it is precisely the same permutation that maps start points to end points in the lattice path representations of tilings of both $\check{H}_{n,2m}^{L,R}$ and $\widehat{H}_{n,2m}^{L,R}$, and so the determinants of $\check{E}_{R,L}$ and $\widehat{E}_{R,L}$ have the same sign.

Theorem 8. *The absolute values of the determinants of the matrices $\check{E}_{R,L}$ and $\widehat{E}_{R,L}$ are both bounded above by 1 for all n, m, R, L where $-n \leq r, l \leq n$ for all $r \in R$ and $l \in L$.*

Proof. In order to prove the above theorem it is sufficient to construct a map

$$\zeta : T_{n,2m}^{R,L} \rightarrow T_{n,2m}^{\emptyset,\emptyset}$$

that maps distinct tilings of $\check{H}_{n,2m}^{L,R}$ to distinct tilings of $\check{H}_{n,2m}^{\emptyset,\emptyset}$.

Suppose first that $|R| = |L| = 1$ and consider the two possible configurations of holes h_1 and h_2 along the zig-zag boundary of $\check{H}_{n,2m}^{L,R}$ (where h_1 is the leftmost unit hole, and h_2 the rightmost, see Figure 10). Beginning at h_1 , one can construct what shall be referred to as a *propagation path* from h_1 to h_2 in the following way:

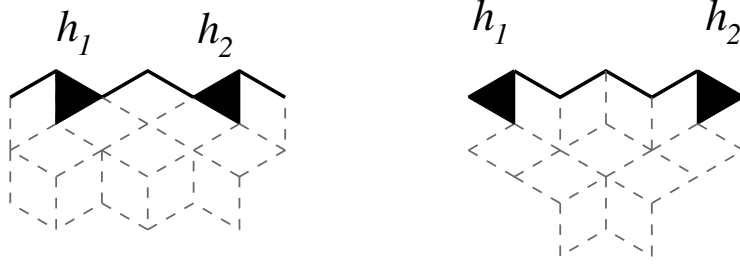


FIGURE 10. The two types of unit hole configuration that can occur within a peak along the zig-zag boundary.

- (i) if h_1 is left pointing then construct a path across unit rhombi that begins at the midpoint of the vertical edge of h_1 and ends at the midpoint of the vertical edge of h_2 ;
- (ii) otherwise consider the path across unit rhombi that begins at the midpoint of the southeast edge of h_1 and ends at the midpoint of the southeast edge of a unit rhombus that lies along the southeast edge of $\check{H}_{n,2m}^{L,R}$. Construct a similar path that begins at the midpoint of the southwest edge of h_2 and ends at the southwest edge of $\check{H}_{n,2m}^{L,R}$.

It should be clear that one may always apply one of the above two steps. In the first case such a path certainly exists since in the translation of tilings to lattice paths, every tiling of $\check{H}_{n,2m}^{L,R}$ corresponds to a family of non-intersecting paths, one of which will begin at h_1 and end at h_2 . In the second case, by translating a tiling to a set of paths across unit rhombi that begin at the southeast edge of $\check{H}_{n,2m}^{L,R}$ one sees that there will always exist a path across rhombi from the southeast edge of $\check{H}_{n,2m}^{L,R}$ that ends at h_1 . Likewise, by translating the same tiling into paths across rhombi that instead begin at the southwest edge of $\check{H}_{n,2m}^{L,R}$, one sees that there will always exist a path across unit rhombi that ends at h_2 . Note that these two paths will intersect at precisely one point, say $p_{1,2}$ (alternatively one could also say that the two paths across rhombi have in common precisely one unit rhombus).

The *propagation path* from h_1 to h_2 is then the ribbon of unit rhombi that lie along the path from h_1 to h_2 , either obtained directly from the path described by case (i), or in the second case by travelling from h_1 to h_2 via the unique intersection point $p_{1,2}$ (see Figure 11).

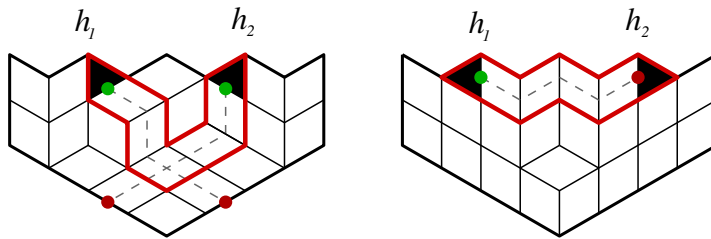


FIGURE 11. The propagation paths (outlined in red) between different configurations of holes h_1 and h_2 .

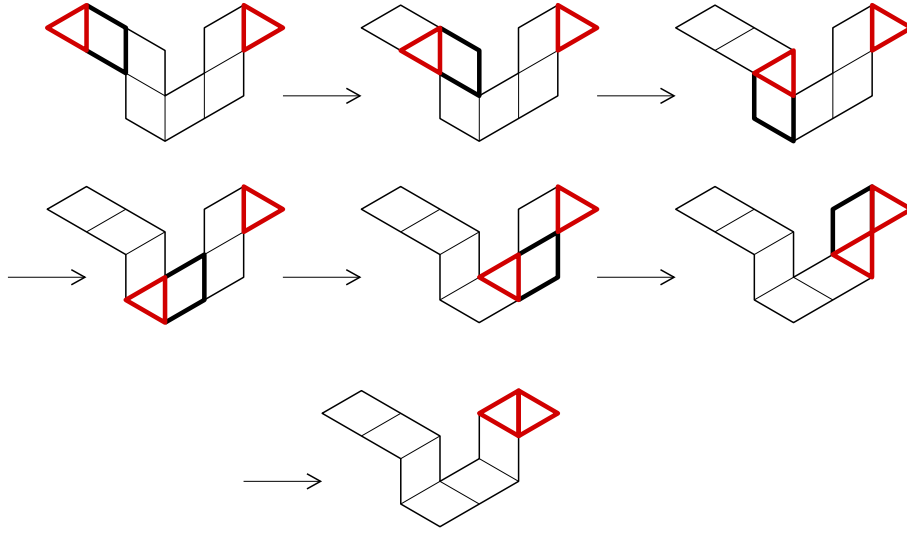


FIGURE 12. The transmission of a left pointing hole along a ribbon consisting of 6 unit rhombi, yielding a ribbon of 7 unit rhombi that contains no unit triangles.

The goal now is to transmit the unit triangle h_1 along the propagation path until it is in a position such that it shares an edge with the unit triangle h_2 . One does this by successively interchanging the unit triangle h_1 with the neighbouring unit rhombus that lies between h_1 and h_2 .

This interchange is defined in the the following way: the unit rhombus that shares an edge with the unit triangle comprises a trapezium, so in order to interchange the two shapes one shifts the unit triangle one unit length along the longest edge of the trapezium. Such an interchange is demonstrated in the following figure:



(one may also see this as a reflection through a certain line of symmetry of the trapezium).

Given a propagation path that contains an arrangement of $k - 1$ unit rhombi, where the ends of the ribbon are unit triangles, one may apply the interchange operation defined above $k - 1$ times in order to transmit the leftmost hole h_1 along the path to the rightmost hole h_2 . In doing so one transforms the original ribbon consisting of two triangular holes and $k - 1$ rhombi into a ribbon of precisely the same shape, consisting instead of k unit rhombi (the pair of triangular holes h_1 and h_2 that share an edge forms a hole that may be equivalently thought of as a unit rhombus). An example of the transmission of a hole along a ribbon of rhombi may be found in Figure 12. It should be clear that transmitting a triangular hole in this way only affects the rhombi that lie along the propagation path, thus any two distinct tilings of $\check{H}_{n,2m}^{L,R}$ may be transformed under such an operation into two distinct tilings of $\check{H}_{n,2m}^{\emptyset,\emptyset}$.

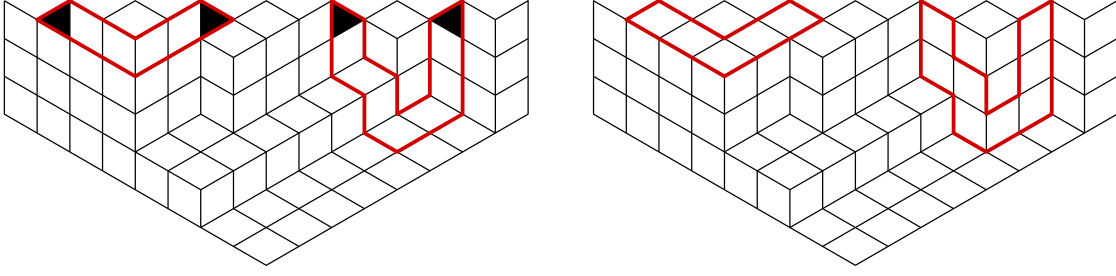


FIGURE 13. A tiling of $\check{H}_{8,6}^{\{-6,6\},\{-2,2\}}$ (left) and its corresponding unholey tiling of $\check{H}_{8,6}^{\emptyset,\emptyset}$ under the map ζ (right). The sets $R = \{-2, 2\}$ and $L = \{-6, 6\}$ give rise to the tuple $(-6, -2, 2, 6)$, from which one obtains the set of ordered (and coloured) pairs of holes $\{(-6, -2), (2, 6)\}$ that determine the propagation paths along which one must apply ζ .

The above argument establishes that the individual entries of the matrix $\check{E}_{R,L}$ are bounded above by 1, however one may easily extend such a mapping so that distinct tilings of $\check{H}_{n,2m}^{L,R}$ containing a finite number of left and right pointing unit triangular holes may be transformed into distinct tilings of $\check{H}_{n,2m}^{\emptyset,\emptyset}$.

Given two sets R and L (where $|R| = |L| = k$) corresponding to multiple right and left pointing unit triangular holes along the zig-zag boundary of $\check{H}_{n,2m}^{L,R}$, colour all the elements of R in one colour and all the elements of L in another. Consider the tuple T obtained by ordering all elements of $R \cup L$ in increasing order (irrespective of colour). From this tuple select the first pair of consecutive elements of differing colours that occur when T is read from left to right, say h_1 and h_2 . Remove these elements from T and form a new tuple consisting of $2k - 2$ coloured elements from $R \cup L \setminus \{h_1, h_2\}$, arranged in increasing order. Repeatedly applying this process (k times in total) yields a set of ordered (and coloured) pairs $\{(h_1, h_2), \dots, (h_{2k-1}, h_{2k})\}$ that determine the holes between which one should construct propagation paths. A moment's thought convinces oneself that selecting pairs of holes in this way ensures that no two distinct propagation paths between pairs of holes intersect, thus it is straightforward to see that under this extended mapping any two distinct tilings of $\check{H}_{n,2m}^{L,R}$ give rise to two distinct tilings of $\check{H}_{n,2m}^{\emptyset,\emptyset}$. Figure 13 illustrates the unholey tiling obtained from a tiling of $\check{H}_{8,6}^{\{-6,6\},\{-2,2\}}$.

What of a map between weighted tilings of $\hat{H}_{n,2m}^{L,R}$ and weighted tilings of $\hat{H}_{n,2m}^{\emptyset,\emptyset}$? Observe that the holes that lie along the zig-zag boundary of $\hat{H}_{n,2m}^{L,R}$ are trapezia that lie within the peaks of the boundary, and such holes are equivalent to fixing a unit triangular hole within a “fold” of the boundary, thus forcing a rhombus that shares with it a vertical edge. One may therefore apply precisely the same map ζ to tilings of $\hat{H}_{n,2m}^{L,R}$ to obtain tilings of $\hat{H}_{n,2m}^{\emptyset,\emptyset}$. At first it would appear that one must be careful to consider the extra weighting of certain configurations, however after a little consideration it is straightforward to see that by transforming a weighted tiling of $\hat{H}_{n,2m}^{L,R}$ into a weighted tiling of $\hat{H}_{n,2m}^{\emptyset,\emptyset}$, one only ever increases the number of possible configurations of tiles that carry a combined weight of 2. Thus the weight of a tiling of $\hat{H}_{n,2m}^{L,R}$ is at most the

weight of its image under ζ , hence the weighted count of all tilings of $\check{H}_{n,2m}^{L,R}$ is at most the weighted count of all tilings of $\check{H}_{n,2m}^{\emptyset,\emptyset}$. \square

Corollary 9. *The number of tilings of $H_{n,2m}^{L,R}$ is at most the number of tilings of $H_{n,2m}$.*

Remark 10. This corollary follows immediately from Theorem 8, however it is in fact a special case of a more general (conjectured) result, chiefly that the number of tilings of a hexagon that contains any amount of unit triangular holes distributed anywhere within its interior is at most the number of tilings of the same region without holes. An injective proof of this result will be the subject of future work by the author.

4. ASYMPTOTICS

The goal of this final section is to establish asymptotic expressions for $\det \widehat{E}_{R,L}$ and $\det \check{E}_{R,L}$ as the size of the boundaries of $\widehat{H}_{n,2m}^{L,R}$ and $\check{H}_{n,2m}^{L,R}$ tend to infinity and the distances between the holes grows large.

Suppose ξ is a positive real number such that $2m \sim \xi n$. For a set of triangular holes determined by R and L within a sea of unit rhombi, the interaction between them is defined to be

$$\omega_H(R, L; \xi) = \lim_{n \rightarrow \infty} \frac{M(H_{n,\xi n}^{L,R})}{M(H_{n,\xi n})},$$

according to (2.1). Inserting the result from Theorem 3 into the right hand side above gives

$$\omega_H(R, L; \xi) = \lim_{n \rightarrow \infty} (\det \check{E}_{R,L} \cdot \det \widehat{E}_{R,L}),$$

where $\check{E}_{R,L}$ and $\widehat{E}_{R,L}$ are matrices of size p , independent of n . According to Theorem 8 the product of these determinants is bounded above by 1 for all n , thus one may consider the limit of the individual entries of each matrix as n tends to infinity. To that end, consider a single entry of $\check{E}_{R,L}$ for $r_j > l_i$,

$$\begin{aligned} \lim_{n \rightarrow \infty} & \left({}_4F_3 \left[\begin{matrix} \frac{r}{2} - \frac{n}{2} + 1, 1, \frac{r}{2} - \frac{l}{2} + 2, \frac{n}{2} + \frac{r}{2} + \frac{1}{2} \\ m + \frac{n}{2} + \frac{r}{2} + 2, \frac{r}{2} - m - \frac{n}{2} + 2, \frac{r}{2} - \frac{l}{2} + \frac{3}{2} \end{matrix}; 1 \right] \right. \\ & \times \frac{\Gamma(m+n+1)\Gamma(\frac{n}{2} + \frac{r_j}{2} + \frac{1}{2})\Gamma(\frac{l_i}{2} + m + \frac{n}{2})\Gamma(m + \frac{n}{2} - \frac{r_j}{2} - 1)}{\Gamma(\frac{n}{2} - \frac{r_j}{2})\Gamma(m - \frac{l_i}{2} + \frac{n}{2} + 1)\Gamma(m + \frac{n}{2} + \frac{r_j}{2} + 2)} \\ & \left. \times \frac{2^{r_j-l_i+2}\Gamma(m + \frac{3}{2})\Gamma(\frac{n}{2} - \frac{l_i}{2} + \frac{1}{2})}{\pi(r_j - l_i + 1)\Gamma(m)\Gamma(\frac{l_i}{2} + \frac{n}{2})\Gamma(m + n - \frac{1}{2})} \right). \end{aligned}$$

Once again by Theorem 8 the above expression is bounded above by 1 for all n . Moreover since the sum is not alternating and terminates one may safely interchange the limit and sum operations as n tends to infinity. Applying Stirling's approximation to the pre-factor above yields

$$\frac{\xi^{3/2}(\xi+2)^{3/2}2^{r_j-l_i+2}}{\pi(r_j-l_i+1)(\xi+1)^{r_j-l_i+4}},$$

while the ${}_4F_3$ hypergeometric series reduces to

$${}_2F_1 \left[\begin{matrix} 1, \frac{r_j}{2} - \frac{l_i}{2} + 2 \\ \frac{r_j}{2} - \frac{l_i}{2} + \frac{3}{2} \end{matrix}; \frac{1}{(\xi+1)^2} \right].$$

As the distance between the holes grows (that is, the distance between the holes at r_j and l_i grows very large), this series reduces to a geometric series, hence

$$\check{e}_{i,j} \sim \frac{(\xi(\xi+2))^{1/2}}{\pi(r_j - l_i)} \left(\frac{2}{\xi+1} \right)^{r_j - l_i + 2}$$

for holes that are far apart and point away from each other in a sea of unit rhombi.

Similarly if the holes point toward each other (that is, if $r_j < l_i$) then as n tends to infinity one obtains

$$\check{e}_{i,j} \sim -\frac{2^{r_j - l_i + 2} (\xi(\xi+2))^{3/2}}{3\pi} \cdot {}_2F_1 \left[2 - \frac{l_i}{2} + \frac{r_j}{2}, \frac{3}{2}; -\xi(\xi+2) \right]. \quad (4.1)$$

Since the above hypergeometric series is terminating one may apply the following transformation formula

$${}_2F_1 \left[a, -n; c; z \right] = \frac{(1-z)^n (a)_n}{(c)_n} {}_2F_1 \left[-n, c-a; 1-a-n; (1-z)^{-1} \right]$$

(see [21, (1.8.10), with sum reversed on the right hand side]) to (4.1), yielding

$$\check{e}_{i,j} \sim \frac{2^{r_j - l_i + 2} (\xi(\xi+2))^{3/2}}{\pi(r_j - l_i + 1)} {}_2F_1 \left[1, \frac{r_j}{2} - \frac{l_i}{2} + 2; \frac{1}{(\xi+1)^2} \right].$$

This agrees completely with the asymptotic approximation of $\check{e}_{i,j}$ for $r_j > l_i$, thus it follows that each entry of the matrix $\check{E}_{R,L}$ is asymptotically

$$\check{e}_{i,j} \sim \frac{(\xi(\xi+2))^{1/2}}{\pi(r_j - l_i)} \left(\frac{2}{\xi+1} \right)^{r_j - l_i + 2},$$

and a similar line of reasoning shows that each entry of $\hat{E}_{R,L}$ is asymptotically

$$\hat{e}_{i,j} \sim \frac{(\xi(\xi+2))^{-1/2}}{\pi(r_j - l_i)} \left(\frac{2}{\xi+1} \right)^{r_j - l_i + 2}.$$

For $\xi \neq 1$ the interaction between holes either blows up or shrinks exponentially according to how they are positioned. More precisely, suppose at first that $\xi > 1$. If the leftmost triangular hole lying within tilings of the plane is right pointing then as the distance between the holes becomes large the entries of an entire column of $\hat{E}_{R,L}$ shrink exponentially, thus the determinant of $\hat{E}_{R,L}$ also shrinks. If instead the leftmost hole is a left pointing triangle then the entries of an entire row blow up exponentially, thus the absolute value of the determinant grows exponentially large. It is easy to see that the same can be said of the matrix $\check{E}_{R,L}$, and similarly if $\xi < 1$ then the converse is true.

Suppose now that $\xi = 1$ and consider the (i, j) -entry of $\check{E}_{R,L}$, that is,

$$\frac{1}{2\pi(x_i - y_j)},$$

where $x_i = -\frac{\sqrt{3}}{2}l_i$ and $y_j = -\frac{\sqrt{3}}{2}r_j$. It is clear that the determinant of $\check{E}_{R,L}$ is the determinant of a certain Cauchy matrix, hence it has the following explicit evaluation

$$|\det \check{E}_{R,L}| = \left(\frac{1}{2\pi} \right)^p \frac{\prod_{i=2}^p \prod_{j=1}^{i-1} d(r_i, r_j) d(l_i, l_j)}{\prod_{1 \leq i, j \leq p} d(r_i, l_j)},$$

where $d(x, y) = \frac{\sqrt{3}}{2}|x - y|$ is the Euclidean distance between the midpoints of the vertical edges of the triangular holes at lattice distances x and y from the origin.

Since the charge of each triangular hole is ± 2 , the right hand side of the above expression may be re-written as

$$\left(\frac{1}{2\pi}\right)^p \prod_{1 \leq j < i \leq |R \cup L|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)}, \quad (4.2)$$

where $h_i, h_j \in R \cup L$. While this expression certainly holds for interleaving triangular holes of side length two it may be refined further in order to express the interaction as a product over holes of any even size.

Suppose R and L each contain a string of contiguous triangular holes r_1, \dots, r_s and l_1, \dots, l_t (respectively) that induce holes h_1 and h_2 of side lengths $2s$ and $2t$ (respectively) as described in Remark 2. The charge of an induced hole is simply the sum of the charges of the holes of side length two that induce it (equivalently, the charge of each hole is \pm twice the number of holes that induce it). As the distance between h_1 and h_2 grows large the distances between the individual triangular holes r_1, \dots, r_s remain constant (similarly for l_1, \dots, l_t), thus

$$\frac{\prod_{i=2}^s \prod_{j=1}^{i-1} d(r_i, r_j) d(l_i, l_j)}{\prod_{i=1}^s \prod_{j=1}^t d(r_i, l_j)} \sim \left(\prod_{h \in \{h_1, h_2\}} \prod_{i=0}^{\frac{1}{2}|q(h)|-1} 3^{i/2} \Gamma(i+1) \right) \cdot d(h_1, h_2)^{\frac{1}{4}q(h_1)q(h_2)}.$$

Letting \mathcal{H} index the set of triangular holes induced by R and L it follows from the above observation that (4.2) may be re-written as

$$\prod_{h \in \mathcal{H}} \left(\prod_{s=0}^{\frac{1}{2}|q(h)|-1} \frac{3^{s/2} \Gamma(s+1)}{\sqrt{2\pi}} \right) \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)}, \quad (4.3)$$

for $h_i, h_j \in \mathcal{H}$, thereby establishing Theorem 2.

Similar arguments may be used to show that

$$|\det \check{E}_{R,L}| \sim \prod_{h \in \mathcal{H}} \left(\prod_{s=0}^{\frac{1}{2}|q(h)|-1} \frac{3^{(s+1)/2} \Gamma(s+1)}{\sqrt{2\pi}} \right) \prod_{1 \leq j < i \leq |\mathcal{H}|} d(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)}.$$

The product of the above expression together with (4.3) then yields Theorem 1.

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